

On the combinatorics of hypergeometric functions

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Abstract

We give combinatorial interpretation for hypergeometric functions associated with tuples of rational numbers.

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1 Introduction

Given a natural number $n \in \mathbb{N}$ and a real number $a \in \mathbb{R}$ the Pochhammer symbol $(a)_n$ is defined by the identity $(a)_n = a(a+1)(a+2)\dots(a+n-1)$. Later we shall also need a generalized form of the Pochhammer symbol, introduced in [5], defined as follows: let $k \in \mathbb{R}$ then

$$(a)_{n,k} = a(a+k)(a+2k)\dots(a+(n-1)k).$$

Now given a tuple of complex numbers a_1, \dots, a_k and another tuple of complex numbers b_1, \dots, b_l such that no b_i is a negative integer or zero, then there is an associated hypergeometric series $h(a_1, \dots, a_k; b_1, \dots, b_l) \in \mathbb{C}[[x]]$, the formal power series with complex coefficients defined by:

$$h(a_1, \dots, a_k; b_1, \dots, b_l) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_k)_n}{(b_1)_n \dots (b_l)_n} \frac{x^n}{n!}.$$

The goal of this paper is to find a combinatorial interpretation of the coefficients $\frac{(a_1)_n \dots (a_k)_n}{(b_1)_n \dots (b_l)_n}$ of the hypergeometric series $h(a_1, \dots, a_k; b_1, \dots, b_l)$. Without further restrictions this problem seems hopeless since in general the coefficients $\frac{(a_1)_n \dots (a_k)_n}{(b_1)_n \dots (b_l)_n}$ are complex numbers and, to this day, no good definition of "combinatorial interpretation" for complex numbers is known.

If we demand that a_1, \dots, a_k and b_1, \dots, b_l be rational numbers, then clearly the coefficients $\frac{(a_1)_n \dots (a_k)_n}{(b_1)_n \dots (b_l)_n}$ will also be rational numbers. In [3] we proposed a general setting in which the phrase "combinatorial interpretation of a rational number" is well defined and makes good sense. Indeed a combinatorial interpretation of Bernoulli and Euler numbers is provided in [3]. So working within that setting the question of the combinatorial interpretation of the coefficients of hypergeometric functions associated with tuples of rational numbers makes sense.

The rest of this paper is devoted to, first, explaining what do we mean when we talk about combinatorial interpretations of rational numbers, and second, use this notion to uncover the combinatorial meaning of the hypergeometric series $h(a_1, \dots, a_k; b_1, \dots, b_l)$ with $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{Q}^+$. Notice that we are restricting our attention to positive rational numbers. We do this for simplicity, the case of negative rational numbers can also be dealt with our methods, and will be the subject of future work.

2 Rational Combinatorics

In this section we explain what we mean when we talk about the combinatorial interpretation of a formal power series in $\mathbb{Q}^+[[x]]$. We only give the material needed to attack the problem of this paper, namely, the combinatorial interpretation of $h(a_1, \dots, a_k; b_1, \dots, b_l)$ with $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{Q}^+$. The interested reader may find more information in references [1] [2], [3] and [4].

Let us define what we mean by the combinatorial interpretation of a positive rational number. To do so we start by introducing the category *gpd* of finite groupoids. Recall that a category C consists of a collection of objects $Ob(C)$, and a set of morphisms $C(x, y)$ for each pair of objects $x, y \in C$, together with a natural list of axioms [12]. Let us recall the definition of a finite groupoid [1].

Definition 1. *A groupoid G is a category such that all its morphisms are invertible. A groupoid G is called finite if $Ob(G)$ is a finite set and $G(x, y)$ is a finite set for all $x, y \in Ob(G)$.*

Definition 2. *The category *gpd* is such that its objects $Ob(gpd)$ are finite groupoids. For groupoids $G, H \in Ob(gpd)$ morphisms in $gpd(G, H)$ are functors $F : G \rightarrow H$.*

There are two rather simple examples of a finite groupoid: 1) Any finite set x may be regarded as a groupoid. The objects of x is x itself and the only morphisms in x are the identities. 2) Any finite group G may be regarded as a groupoid \overline{G} . \overline{G} has only one object 1 and $\overline{G}(1, 1) = G$.

The category *gpd* has several remarkable properties which we briefly summarize:

- There is a bifunctor $\sqcup : gpd \times gpd \longrightarrow gpd$ called disjoint union and defined for G, H in *gpd* by $Ob(G \sqcup H) = Ob(G) \sqcup Ob(H)$ and for $x, y \in Ob(G \sqcup H)$ the morphisms from x to y are given by

$$G \sqcup H(x, y) = \begin{cases} G(x, y) & \text{if } x, y \in Ob(G), \\ H(x, y) & \text{if } x, y \in Ob(H), \\ \emptyset & \text{otherwise.} \end{cases}$$

- There is a bifunctor $\times : gpd \times gpd \longrightarrow gpd$ called Cartesian product and defined for G, H in *gpd* by $Ob(G \times H) = Ob(G) \times Ob(H)$ and for all $(x_1, y_1), (x_2, y_2) \in Ob(G \times H)$ we have $G \times H((x_1, y_1), (x_2, y_2)) = G(x_1, x_2) \times H(y_1, y_2)$.
- There is an empty groupoid \emptyset . It is such that $Ob(\emptyset) = \emptyset$. \emptyset is the neutral element in *gpd* with respect to disjoint union.

- There is a groupoid 1 such that it has only one object and only one morphism, the identity, between that object and itself. 1 is an identity with respect to Cartesian product in gpd .
- There is a valuation map $|\cdot| : Ob(gpd) \rightarrow \mathbb{Q}^+$, given on $G \in Ob(gpd)$ by

$$|G| = \sum_{x \in D(G)} \frac{1}{|G(x, x)|},$$

where $D(G)$ denotes the set of isomorphisms classes of objects of G and for $|A|$ denotes the cardinality of A . $|G|$ is called the cardinality of the groupoid G .

- The valuation map $|\cdot| : Ob(gpd) \rightarrow \mathbb{Q}^+$ is such that $|G \sqcup H| = |G| + |H|$, $|G \times H| = |G||H|$ for all $G, H \in gpd$. Also $|\emptyset| = 0$ and $|1| = 1$.

The properties above suggest the following

Definition 3. If $a = |G|$, where $a \in \mathbb{Q}^+$ and $G \in gpd$, then we say that a is the cardinality of the groupoid G , and also that G is a combinatorial interpretation (in terms of finite groupoids) of the rational number a .

Notice that there are many groupoids with the same cardinality, so the combinatorial interpretation of a rational number is by no means unique. This should not be seen as something negative, indeed the freedom to choose among several different possible interpretations is what makes the subject of enumerative combinatorics (either for integers or for rationals) so fascinating. For example the reader may look at [13] where a list of references with over 30 different combinatorial interpretations of the Catalan numbers is given.

For example, in this paper we choose the combinatorial interpretation of the number $\frac{1}{n}$ to be $\overline{\mathbb{Z}}_n$ the groupoid associated to the cyclic group with n -elements. We believed our choice is justified for simplicity. Nevertheless, this seemingly naive choice is actually quite subtle. For example, we think that the number $\frac{1}{nm}$ is better interpreted by $\overline{\mathbb{Z}}_n \times \overline{\mathbb{Z}}_m$ than by $\overline{\mathbb{Z}}_{nm}$. Since $\overline{\mathbb{Z}}_n \times \overline{\mathbb{Z}}_m$ and $\overline{\mathbb{Z}}_{nm}$ are not isomorphic groupoids they provide really different combinatorial interpretations for $\frac{1}{nm}$. Notice that in general we have that $|\overline{\mathbb{Z}}_b^{\oplus a}| = \frac{a}{b}$.

Now that we have defined what we mean by a combinatorial interpretation (in terms of finite groupoids) of a rational number, we proceed to define what is a combinatorial interpretation of a formal power series with positive rational coefficients. First we introduce the groupoid \mathbb{B} . Objects of \mathbb{B} are finite sets. For finite sets x and y , morphisms in $\mathbb{B}(x, y)$ are bijections $f : x \rightarrow y$.

Definition 4. The category of \mathbb{B} -gpd species is the category $gpd^{\mathbb{B}}$ of functors from \mathbb{B} to gpd . An object of $gpd^{\mathbb{B}}$ is called a \mathbb{B} -gpd species or a species.

The category of rational species $gpd^{\mathbb{B}}$ has the following properties

- There is a bifunctor $+\ : gpd^{\mathbb{B}} \times gpd^{\mathbb{B}} \longrightarrow gpd^{\mathbb{B}}$ given for $F, G \in gpd^{\mathbb{B}}$ as follows: $F+G(x) = F(x) \sqcup G(x)$ for all $x \in Ob(\mathbb{B})$. Bifunctor $+$ is called the sum of species.

- There is a bifunctor $\times : gpd^{\mathbb{B}} \times gpd^{\mathbb{B}} \longrightarrow gpd^{\mathbb{B}}$ given for $F, G \in gpd^{\mathbb{B}}$ as follows $F \times G(x) = F(x) \times G(x)$ for all $x \in Ob(\mathbb{B})$. The bifunctor \times is called the Hadamard product.
- There is a bifunctor $gpd^{\mathbb{B}} \times gpd^{\mathbb{B}} \longrightarrow gpd^{\mathbb{B}}$ given for $F, G \in gpd^{\mathbb{B}}$ as follows: $FG(x) = \bigsqcup_{a \subseteq x} F(a) \times G(x \setminus a)$ for all $x \in Ob(\mathbb{B})$. This bifunctor is called the product of species.
- Let $gpd_0^{\mathbb{B}}$ be the full subcategory of species F such that $F(\emptyset) = \emptyset$. There is a bifunctor

$$\circ : gpd^{\mathbb{B}} \times gpd_0^{\mathbb{B}} \longrightarrow gpd^{\mathbb{B}}$$

given for $F \in gpd^{\mathbb{B}}$ and $G \in gpd_0^{\mathbb{B}}$ by

$$F \circ G(x) = \bigsqcup_{p \in \Pi(x)} F(p) \prod_{b \in p} G(b)$$

for all $x \in Ob(\mathbb{B})$. Above $\Pi(x)$ denotes the set of partitions of x . This bifunctor is called the composition or substitution of species.

- The species 1 is defined by $1(x) = 1$ if $|x| = 1$ and $1(x) = \emptyset$ otherwise. The species 0 is defined by $0(x) = \emptyset$ if $x \neq \emptyset$ and $0(\emptyset) = 1$.
- There is a valuation map $|\cdot| : Ob(gpd^{\mathbb{B}}) \rightarrow \mathbb{Q}^+[[x]]$ given on $F \in Ob(gpd^{\mathbb{B}})$ by

$$|F| = \sum_{n \in \mathbb{N}} |F[n]| \frac{x^n}{n!},$$

where $[n] = \{1, 2, \dots, n\}$ and $[0] = \emptyset$.

- The valuation map $|\cdot| : Ob(gpd^{\mathbb{B}}) \rightarrow \mathbb{Q}^+[[x]]$ satisfies the following properties: $|F + G| = |F| + |G|$, $|F \times G| = |F| \times |G|$, the Hadamard product of powers series, i.e., coefficientwise multiplication, $|FG| = |F||G|$, $|F \circ G| = |F|(|G|)$, $|1| = 1$, and $|0| = 0$.

The properties above allow one to make the following

Definition 5. If $f = |F|$, where $f \in \mathbb{Q}^+[[x]]$ and $F \in Ob(gpd^{\mathbb{B}})$, then we say that f is the generating series associated with the species F , and also that the species F is a combinatorial interpretation (in terms of functors from finite sets to groupoids) of the formal power series f .

For example let $\mathbb{Z} : \mathbb{B} \rightarrow gpd$ be the species sending $x \in Ob(\mathbb{B})$ into the groupoid $\mathbb{Z}(x)$ such that $Ob(\mathbb{Z}(x)) = \begin{cases} \{x\} & \text{if } x \neq \phi, \\ \phi & \text{if } x = \phi. \end{cases}$ and $\mathbb{Z}(x)(x, x) = \mathbb{Z}_{|x|}$. Then

$$|\mathbb{Z}| = \sum_{n=0}^{\infty} |\mathbb{Z}(n)| \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{n!} = \int \frac{e^x - 1}{x} dx.$$

After we have made all these remarks we can rephrase our original problem of finding a combinatorial interpretation for the hypergeometric series as follows: given tuples of positive rational numbers $a_1, \dots, a_k, b_1, \dots, b_l$, find a species $H(a_1, \dots, a_k; b_1, \dots, b_l) : \mathbb{B} \rightarrow gpd$ such that $|H(a_1, \dots, a_k; b_1, \dots, b_l)| = h(a_1, \dots, a_k; b_1, \dots, b_l)$. A solution to this problem will be given in

the next section. We remark that at this point our extension of the notion of combinatorial species introduced by Joyal in [7] and [8], replacing the target category from \mathbb{B} to gpd becomes necessary. The generating series associated with a combinatorial species is a formal power series with integer coefficients, and thus in that formalisms there is no hope to find functors $H(a_1, \dots, a_k; b_1, \dots, b_l)$ with the desired properties.

3 Combinatorial interpretation of hypergeometric functions

In this section we construct functors

$$H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right) : \mathbb{B} \rightarrow gpd$$

for $a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_l, d_1, \dots, d_l \in \mathbb{N}^+$ such that

$$|H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)| = h\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right).$$

The construction of functors $H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)$ provides the combinatorial interpretation for the hypergeometric functions $h\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)$ promised in the introduction. As remarked above combinatorial interpretations are by no means unique. Nevertheless they are not arbitrary, and a good combinatorial interpretation is subject to many constraints. In our construction of $H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)$ we are going to carefully respect the Hadamard product structure implicit in the definition of $h\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)$. Namely, since

$$h\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right) = \times_{i=1}^k h\left(\frac{a_i}{b_i}; \emptyset\right) \times \times_{j=1}^l h\left(\emptyset, \frac{c_j}{d_j}\right),$$

it is natural to demand that

$$H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right) = \times_{i=1}^k H\left(\frac{a_i}{b_i}; \emptyset\right) \times \times_{j=1}^l H\left(\emptyset, \frac{c_j}{d_j}\right),$$

where $|H\left(\frac{a_i}{b_i}; \emptyset\right)| = h\left(\frac{a_i}{b_i}; \emptyset\right)$ and $|H\left(\emptyset, \frac{c_j}{d_j}\right)| = h\left(\emptyset, \frac{c_j}{d_j}\right)$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$. We begin by introducing some definitions and notations. Next we introduce the analogue in the category gpd of the generalized Pochhammer symbol.

Definition 6. Fix a finite groupoid K . The functorial Pochhammer symbol $(\)_{n,K} : gpd \rightarrow gpd$ is given by

$$(G)_{n,K} = \prod_{i=0}^{n-1} (G \sqcup (K \times [i])),$$

for G in gpd . If $K = 1$ we write $(\)_n$ instead of $(\)_{n,1}$.

With this notation we have the following

Lemma 7. If $|K| = k$ and $|G| = g$, then $|(G)_{n,K}| = (g)_{n,k}$.

Proof. $|(G)_{n,K}| = |\prod_{i=0}^{n-1} (G \sqcup (K \times [i]))| = \prod_{i=0}^{n-1} (|G| + |K|) = (g)_{n,k}$. □

For example one can let the set $Ob((\overline{\mathbb{Z}}_b^{\oplus a})_n)$ be $[a] \times [a+1] \times \dots \times [a+n-1]$. An object $x \in Ob((\overline{\mathbb{Z}}_b^{\oplus a})_n)$ is a tuple $x = (x_1, \dots, x_n)$ such that $x_i \in \mathbb{N}$ and $1 \leq x_i \leq a+i-1$. There are morphisms only from an object to itself and

$$(\overline{\mathbb{Z}}_b^{\oplus a})_n(x, x) = \overline{\mathbb{Z}}_b^{\times c(x)},$$

where $c(x) = |\{i \mid 1 \leq x_i \leq a\}|$ for $x \in Ob((\overline{\mathbb{Z}}_b^{\oplus a})_n)$. Clearly $|(\overline{\mathbb{Z}}_b^{\oplus a})_n| = (\frac{a}{b})_n$.

Definition 8. For $a, b \in \mathbb{N}^+$ we let the functor $H(\frac{a}{b}, \emptyset) : \mathbb{B} \rightarrow \text{gpd}$ be such that on $x \in \mathbb{B}$ it is given by

$$H(\frac{a}{b}, \emptyset)(x) = ([a])_{|x|, [b]} \overline{\mathbb{Z}}_b^x.$$

Explicitly $Ob(H(\frac{a}{b}, \emptyset)(x)) = [a] \times [a+b] \times \dots \times [a+(|x|-1)b]$, and for $s \in Ob(H(\frac{a}{b}, \emptyset)(x))$ we have that $H(\frac{a}{b}, \emptyset)(x)(s, s) = \overline{\mathbb{Z}}_b^x$.

Lemma 9. For $a, b \in \mathbb{N}^+$ we have that $|H(\frac{a}{b}, \emptyset)| = h(\frac{a}{b}, \emptyset) = \frac{1}{(1-x)^{\frac{a}{b}}}$.

Proof. For $n \in \mathbb{N}$ one obtains $|H(\frac{a}{b}, \emptyset)([n])| = |([a])_{n, [b]}| \overline{\mathbb{Z}}_b^{[n]}| = \frac{(a)_{n, b}}{b^n} = (\frac{a}{b})_n$. This identity implies the desired result. \square

Another combinatorial interpretation for $h(\frac{a}{b}, \emptyset)$ is given by

Lemma 10. The functor sending a set x into $(\overline{\mathbb{Z}}_b^{\oplus a})_{|x|}$ provides a combinatorial interpretation for $h(\frac{a}{b}, \emptyset)$.

Definition 11. For $c, n, d \in \mathbb{N}$ we set $\overline{\mathbb{Z}}_{c, n, d} = \prod_{i=0}^{n-1} \overline{\mathbb{Z}}_{c+id}$.

Lemma 12. For $c, n, d \in \mathbb{N}$ we have that $|\overline{\mathbb{Z}}_{c, n, d}| = \frac{1}{(c)_{n, d}}$.

Definition 13. For $c, d \in \mathbb{N}^+$ we let the functor $H(\emptyset; \frac{c}{d}) : \mathbb{B} \rightarrow \text{gpd}$ be such that on $x \in \mathbb{B}$ it is given by

$$H(\emptyset; \frac{c}{d})(x) = [d]^x \overline{\mathbb{Z}}_{c, |x|, d}.$$

Lemma 14. For $c, d \in \mathbb{N}^+$ we have that $|H(\emptyset; \frac{c}{d})| = h(\emptyset; \frac{c}{d})$.

Proof. For $n \in \mathbb{N}$ one obtains $|H(\emptyset; \frac{c}{d})([n])| = |[d]^{[n]} \overline{\mathbb{Z}}_{c, n, d}| = \frac{d^n}{(c)_{n, d}} = (\frac{c}{d})_n$. This identity implies the desired result. \square

We are now ready to define the functor $H(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}) : \mathbb{B} \rightarrow \text{gpd}$ for $a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_l, d_1, \dots, d_l \in \mathbb{N}^+$.

Definition 15. For $x \in \mathbb{B}$ we set

$$H(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l})(x) = \prod_{i=1}^k ([a_i])_{|x|, [b_i]} \overline{\mathbb{Z}}_{b_i}^x \prod_{j=1}^l [d_j]^x \overline{\mathbb{Z}}_{c_j, |x|, d_j}.$$

Theorem 16. $|H(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l})| = h(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l})$.

Proof. It follows from Lemma 7, Lemma 9 and the fact that

$$H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{b_1}{c_1}, \dots, \frac{b_l}{c_l}\right) = \times_{i=1}^k H\left(\frac{a_i}{b_i}, \emptyset\right) \times_{j=1}^l H\left(\emptyset; \frac{c_j}{d_j}\right).$$

Since then

$$|\times_{i=1}^k H\left(\frac{a_i}{b_i}, \emptyset\right) \times_{j=1}^l H\left(\emptyset; \frac{c_j}{d_j}\right)| = \times_{i=1}^k h\left(\frac{a_i}{b_i}, \emptyset\right) \times_{j=1}^l h\left(\emptyset; \frac{c_j}{d_j}\right) = h\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{b_1}{c_1}, \dots, \frac{b_l}{c_l}\right).$$

□

According to Definition 6 the groupoid $H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{b_1}{c_1}, \dots, \frac{b_l}{c_l}\right)(x)$ may be described explicitly as the groupoid whose objects are triples (I, f, g) such that: 1) $I = (I_1, \dots, I_k)$ and $I_i \subseteq [|x| - 1]$. 2) $f = (f_1, \dots, f_k)$ and $f_i : \{0, \dots, |x| - 1\} \rightarrow [a_i] \sqcup ([b_i] \times \mathbb{N})$ is such that $f(i) \in [a_i]$ if $i \notin I$, and if $i \in I$ then $f(i) \in [b_i] \times \mathbb{N}$ and $1 \leq \pi_{\mathbb{N}}(f(i)) \leq i$. 3) $g = (g_1, \dots, g_l)$ where $g_j : x \rightarrow [d_j]$ for $1 \leq j \leq l$. Morphisms are described as follows:

$$H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)(x)((I, f, g), (J, p, q)) = \prod_{i=1}^k \overline{\mathbb{Z}}_{b_i}^{|x|} \prod_{j=1}^l \overline{\mathbb{Z}}_{c_j, |x|, d_j},$$

if $I = J$, $f_i|_{I_i} = p_i|_{I_i}$, and $g_j = q_j$; otherwise,

$$H\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{c_1}{d_1}, \dots, \frac{c_l}{d_l}\right)(x)((I, f, g), (J, p, q)) = \emptyset.$$

Another combinatorial interpretation for $h\left(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_l}; \frac{b_1}{c_1}, \dots, \frac{b_l}{c_l}\right)$ is provided by the functor sending a finite set x into

$$\prod_{i=1}^k (\overline{\mathbb{Z}}_{b_i}^{\oplus a_i})_{|x|} \prod_{j=1}^l [d_j]^x \overline{\mathbb{Z}}_{c_j, |x|, d_j}.$$

We close this paper by stating three open problems. In this work we have provided a combinatorial interpretation for hypergeometric functions, thus the following problem seems natural.

Problem 17. *Find a combinatorial interpretation for the q -hypergeometric functions.*

In [3] we provide a combinatorial interpretation for the Bernoulli and Euler numbers. Our interpretation of the Bernoulli numbers seems to be related to the results of [9].

Problem 18. *Find a combinatorial interpretation for the q -Bernoulli numbers from [11].*

Problem 19. *Find a combinatorial interpretation for the q -Euler numbers.*

The interested reader will find more information on the various q -analogues of the Bernoulli and Euler numbers in references [10], [11].

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References

- [1] John C. Baez, James Dolan. *From finite set to Feynman diagrams*. Mathematics Unlimited - 2001 and Beyond, vol. 1, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50.
- [2] Bergeron, Labelle, and Leroux. *Combinatorial Species and Tree-Like Structures*. Cambridge University Press, Cambridge 1997.
- [3] Héctor Blandín and Rafael Díaz. *Rational combinatorics*. To appear in Advances in Applied Mathematics. arXiv:math.CO/0606041.
- [4] Rafael Díaz and Eddy Pariguan. *Super, quantum and noncommutative species*. arXiv:math.CT/0509674.
- [5] Rafael Díaz and Eddy Pariguan. *On hypergeometric functions and Pochhammer k -symbol*. arXiv:math.CT/0405596.
- [6] Gian-Carlo Rota. *Gian-Carlo Rota on Combinatorics. Introductory papers and commentaries*. Birkhäuser, Boston 1995.
- [7] J. André. *Analytic functors and species of structures*, Lectures Notes in Mathematics, 1234, 126-159, Springer, Berlin 1986.
- [8] J. André. *A combinatorial theory of formal series*. Adv. in Math., 42, 1981, no. 1, 1-82.
- [9] Lee-Chae Jang, Seoung-Dong Kim, Won Park and Soon Ro. *A note on Euler numbers and polynomials*. Journal of Inequalities and Applications. Volume 2006. ID 34602, 5 pages.
- [10] Taeyun Kim. *q -Euler and Genocchi numbers*. Journal of Mathematical Analysis and Applications. In press.
- [11] Taeyun Kim. *Some formulae for the q -Bernoulli and Euler polynomials of higher order*. Journal of Mathematical Analysis and Applications 273 (2002), 236-242.
- [12] S. Mac Lane. *Categories for the Working Mathematician*, vol 5, Graduate Texts in Mathematics, Springer-Verlag, New York, NY, 1971.
- [13] Richard Stanley. *Enumerative Combinatorics*, vol 2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press 2001.

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